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DETERMINATION OF THE CONSTITUTIVE COEFFICIENTS FOR A MIXTURE OF TWO SOLIDS

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Abstract—Using a theory of mixtures allowing different temperatures for each constituent, we first discuss non-linear constitutive equations for a mixture of two elastic solids. Then, upon specialization, we determine relations between certain of the material constants related to each constituent and the corresponding constants related to the mixture. We linearize the constitutive equations by assuming that all displacements from the undeformed state, and all subsequent temperature changes, are small and we further show how to evaluate the constants appearing in the linearized constitutive equations, under certain assumptions which are shown to be thermodynamically consistent.

1. INTRODUCTION

THE theory of a mixture of two elastic solids has been considered by several authors, from various viewpoints (see, e.g. [1]–[6] and the references given therein). In almost all these approaches the deformations of the solids were assumed small and thermodynamical considerations were not included.

The physical motivation for considering such a theory is to attempt to describe the behaviour of certain binary alloys or certain kinds of composite materials.

The major concern when considering a linearized theory of a mixture of two solids is to try to predict, or predict bounds for, the material constants of the mixture from the material constants of each constituent separately, with the help of certain simple experiments. Most of the theories given in references [1]–[6] do not include any interaction terms in the partial stresses, except possibly due to the densities of each constituent.

In this paper we attempt to evaluate the coefficients in a linearized theory of two isotropic elastic solids, from theoretical considerations and with the help of two simple experiments. The linearized constitutive equations which we use contain full interaction terms and we allow small temperature changes.

It is hoped that the theory considered here could be applied to certain heterogeneous composite materials in which each point of the mixture can be considered as being occupied by a particle of each solid when considered on a macroscopic scale, as well as to binary mixtures of continua. In such a material there seems to be no reason why relative motion of the constituents could not occur (cf. Shewmon [5]) and therefore we allow relative motion in this work.

In Section 2 we restate the constitutive equations for a mixture of two non-linear anisotropic elastic solids as derived by Green and Steel [7] using the theory of Green and Naghdi [8], which considers a single temperature distribution for each constituent. We also briefly restate the basic equations of the theory of mixtures given by Green and Naghdi [9] which allows each constituent to have a different temperature. Having done this, we reformulate the constitutive equations for a mixture of two non-linear anisotropic elastic solids in the framework of the latter theory in Section 3, and use the corresponding thermodynamical equations to derive the appropriate restrictions upon these constitutive equations. We thereby obtain the relation between the entropy per unit mass of each constituent and the entropy per unit mass of the mixture in the case when the temperatures of each constituent are equal.

In Section 4 we consider the linearized equations for a mixture of two isotropic elastic solids (defined in a natural manner in Mills and Steel [10]), and use the relation obtained in Section 3 to express certain material coefficients of the mixture in terms of the material coefficients of each constituent in the mixture.

In Section 5 we consider a mixture of two isotropic solids under certain simplifying assumptions but still including interaction terms, and we show how, in principle, the material coefficients occurring in the constitutive equations can be determined by two simple experiments.

2. BASIC EQUATIONS

We use the same notation as reference [9]. All co-ordinates are referred to fixed rectangular Cartesian axes and Latin indices take the values 1, 2, 3. Greek superscripts between parentheses take the values 1, 2 and refer to the constituents 1 or 2 respectively. The summation convention is assumed for repeated Latin indices unless otherwise stated, but any summation over Greek superscripts will be explicitly shown. The positions of typical particles of each constituent at time τ are denoted by $x_i^{(\alpha)}(\tau)$, where

$$x_i^{(\alpha)}(\tau) = x_i^{(\alpha)}(X_1^{(\alpha)}, X_2^{(\alpha)}, X_3^{(\alpha)}, \tau) \qquad (-\infty < \tau \leq t), \tag{2.1}$$

and $X_i^{(\alpha)}$ is a reference position of the particle of the αth constituent.

The density of each particle is denoted by $\rho^{(1)}$ and $\rho^{(2)}$ respectively, and we define velocity vectors $v_i^{(\alpha)}$, acceleration vectors $f_i^{(\alpha)}$, rates of deformation tensors $d_{ik}^{(\alpha)}$, vorticity tensors $\omega_{ik}^{(\alpha)}$ and the time derivatives moving with each particle, $\mathrm{D}^{(\alpha)}/\mathrm{D}t$, in the usual manner (see e.g. Green and Naghdi [9]). We also define a mean velocity w_i and thereby define the time derivative moving with the mixture, $\mathrm{D}/\mathrm{D}t$. The total mass density is given by $\rho = \rho^{(1)}+\rho^{(2)}$.

An important assumption is that the mixture is heterogeneous, that is, each point in it is occupied simultaneously by particles of each constituent. In the present notation, this means that

$$x_i^{(1)}(t) = x_i^{(2)}(t) = x_i, \tag{2.2}$$

and a comma in the following denotes differentiation with respect to x_i.

The constitutive equations for a mixture of two non-linear anisotropic elastic solids in terms of thermodynamic quantities related to the mixture as a whole, are given by†

$$A = A\left(\frac{\partial x_i^{(1)}}{\partial X_j^{(1)}}, \frac{\partial x_m^{(2)}}{\partial X_n^{(2)}}, T\right), \tag{2.3}$$

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† See [7]; the tensors E_{ij} and H_{ij} used there are given by $E_{ij} = F_{ri}^{(1)}F_{rj}^{(1)}$ and $H_{ij} = F_{ri}^{(1)}F_{rj}^{(2)}$.

A tensor function ϕ_{ki}, dependent upon the deformation gradients of each solid, can be added to $\sigma_{ki}^{(1)}$ and subtracted from $\sigma_{ki}^{(2)}$, together with the addition of $\phi_{ki,k}$ to $\bar{\pi}_i^{(1)}$; see Green and Naghdi [11] where it is shown that such additions leave, in fact, all field equations and the entropy inequality unaltered. The essential part of the subsequent work in this paper is therefore unaffected by such additions and we take the constitutive assumption $\phi_{ki} = 0$.

$$S = -\frac{\partial A}{\partial T}, \tag{2.4}$$

$$\sigma^{(\alpha)}_{(ik)} = \tfrac{1}{2}\rho\left(F^{(\alpha)}_{ij}\frac{\partial A}{\partial F^{(\alpha)}_{kj}} + F^{(\alpha)}_{kj}\frac{\partial A}{\partial F^{(\alpha)}_{ij}}\right), \tag{2.5}$$

$$\sigma^{(1)}_{[ki]} = -\sigma^{(2)}_{[ki]} = D_{ki}, \tag{2.6}$$

$$\begin{aligned} D_{ki} &= \tfrac{1}{2}\rho\left(F^{(1)}_{kj}\frac{\partial A}{\partial F^{(1)}_{ij}} - F^{(1)}_{ij}\frac{\partial A}{\partial F^{(1)}_{kj}}\right) \\ &= -\tfrac{1}{2}\rho\left(F^{(2)}_{kj}\frac{\partial A}{\partial F^{(2)}_{ij}} - F^{(2)}_{ij}\frac{\partial A}{\partial F^{(2)}_{kj}}\right), \end{aligned} \tag{2.7}$$

$$\bar{\pi}^{(1)}_k = \rho^{(1)}\frac{\partial A}{\partial F^{(2)}_{ij}}\{\text{grad } F^{(2)}_{ij}\}_k - \rho^{(2)}\frac{\partial A}{\partial F^{(1)}_{ij}}\{\text{grad } F^{(1)}_{ij}\}_k + a_{kj}(v^{(1)}_j - v^{(2)}_j). \tag{2.8}$$

In the above $\sigma^{(\alpha)}_{ik}$ denotes the partial stress in the αth constituent and $\bar{\pi}^{(1)}_k (= -\bar{\pi}^{(2)}_k)$ denotes the diffusive resistance or diffusive force vector.[†] A is the Helmholtz free energy and S the entropy, both per unit mass of the mixture, and T is the temperature of the mixture. $F^{(\alpha)}_{ij}$ is the deformation gradient tensor for the αth constituent, given by

$$F^{(\alpha)}_{ij} = \left(\frac{\partial x^{(\alpha)}_i}{\partial X^{(\alpha)}_j}\right), \tag{2.9}$$

and a_{kj} is a function of the deformation gradients of each constituent and the temperature of the mixture. Also

$$\{\text{grad } F^{(\alpha)}_{ij}\}_k = F^{(\alpha)}_{ij,k} \tag{2.10}$$

and parentheses denote the symmetric part of a tensor, square brackets denote the anti-symmetric part. Equation $(2.7)_2$ can be shown to be identically satisfied using invariance requirements under superposed rigid body motions.

We now briefly list the basic equations that we shall need for a mixture of two chemically inert constituents which are in motion relative to each other and which are allowed to have different temperatures.[†] These are

$$\begin{aligned} -\rho^{(1)}\left[S^{(1)}\frac{DT^{(1)}}{Dt} + \frac{DA^{(1)}}{Dt}\right] + \sigma^{(1)}_{(ki)}d^{(1)}_{ik} - \beta^{(1)}_k(v^{(1)}_k - v^{(2)}_k) \\ - \beta^{(1)}_{ki}(\omega^{(1)}_{ik} - \omega^{(2)}_{ik}) - \frac{\rho^{(1)}q_k T_{,k}}{\rho T} + \Psi^{(1)} - \Phi^{(1)} \geq 0, \end{aligned} \tag{2.11}$$

$$\begin{aligned} -\rho^{(2)}\left[S^{(2)}\frac{DT^{(2)}}{Dt} + \frac{DA^{(2)}}{Dt}\right] + \sigma^{(2)}_{(ki)}d^{(2)}_{ik} - \gamma^{(2)}_k(v^{(2)}_k - v^{(1)}_k) \\ - \gamma^{(2)}_{ki}(\omega^{(2)}_{ik} - \omega^{(1)}_{ik}) - \frac{\rho^{(2)}q_k T_{,k}}{\rho T} + \Psi^{(2)} - \Phi^{(2)} \geq 0, \end{aligned} \tag{2.12}$$

[†] See [9].

where $A^{(\alpha)}$ and $S^{(\alpha)}$ are the Helmholtz free energy and the entropy respectively both per unit mass of the αth constituent, and $T^{(\alpha)}$ is the temperature of the αth constituent. q_k is the heat flux vector per unit area of the mixture per unit time, and $\beta_k^{(1)} = -\beta_k^{(2)}$, $\gamma_k^{(2)} = -\gamma_k^{(1)}$, $\beta_{ki}^{(1)} = -\beta_{ki}^{(2)}$, $\gamma_{ki}^{(2)} = -\gamma_{ki}^{(1)}$, $\Psi^{(\alpha)}$ and $\Phi^{(\alpha)}$ are quantities which occur naturally in the theory. They satisfy the equations

$$\theta_i^{(1)}+\bar{\pi}_i^{(1)}+\gamma_i^{(1)}+\beta_i^{(1)} = 0, \tag{2.13}$$

$$\sigma_{[ki]}^{(1)}+\lambda_{ki}^{(1)}+\gamma_{ki}^{(1)}+\beta_{ki}^{(1)} = 0, \tag{2.14}$$

where $\theta_i^{(1)}(= -\theta_i^{(2)})$ and $\lambda_{ki}^{(1)}(= -\lambda_{ki}^{(2)})$ are the internal force and couple acting on constituent 1 due to interactions (mechanical and thermal). There are two more equations corresponding to these for the second constituent, but these are redundant when there are only two constituents.†

We have already used the rate of workequations for each constituent in writing down (2.11) and (2.12), however we shall not write down the rate of work equations themselves or the equations of motion, as we shall not use them here. For completeness however, the reader is referred to Green and Naghdi [9], equations (4.38), (4.39), (3.11), (4.15) and the boundary conditions are governed by equation (4.32).

The only other equations which we shall need are

$$\begin{aligned}\rho\left(\frac{\mathrm{D}A}{\mathrm{D}t}+S\frac{\mathrm{D}T}{\mathrm{D}t}\right) = \sum_\alpha \Bigg\{&\rho^{(\alpha)}\frac{\mathrm{D}A^{(\alpha)}}{\mathrm{D}t}+\rho^{(\alpha)}S^{(\alpha)}\frac{\mathrm{D}T^{(\alpha)}}{\mathrm{D}t}-\theta_k^{(\alpha)}v_k^{(\alpha)}\\ &-\lambda_{ki}^{(\alpha)}\omega_{ik}^{(\alpha)}+\Phi^{(\alpha)}-\Psi^{(\alpha)}\Bigg\},\end{aligned} \tag{2.15}$$

$$\rho T\frac{\mathrm{D}S}{\mathrm{D}t} = \sum_\alpha\left\{\rho^{(\alpha)}T^{(\alpha)}\frac{\mathrm{D}S^{(\alpha)}}{\mathrm{D}t}-\Phi^{(\alpha)}\right\}. \tag{2.16}$$

We are now in a position to postulate constitutive equations for a mixture of two elastic solids allowing each to have a different temperature.

3. CONSTITUTIVE EQUATIONS

We need to postulate constitutive equations for $A^{(1)}$, $A^{(2)}$, $S^{(1)}$, $S^{(2)}$, $\sigma_{(ik)}^{(1)}$, $\sigma_{(ik)}^{(2)}$, $\beta_k^{(1)}$, $\gamma_k^{(2)}$, $\beta_{ki}^{(1)}$, $\gamma_{ki}^{(2)}$, $\lambda_{ki}^{(1)}$, $\Psi^{(1)}$, $\Psi^{(2)}$, $\Phi^{(1)}$ and $\Phi^{(2)}$. For completeness we should also postulate a constitutive equation for q_k but for what we are interested in here this is not necessary. We also need a constitutive equation for $\bar{\pi}_k^{(1)}$ which we shall consider later, and a constitutive equation for the quantity $[t_i-p_i-n_k(\sigma_{ki}^{(1)}-\sigma_{ki}^{(2)})]$‡ which helps to determine the dynamic boundary conditions needed. Since we will not be considering any dynamic boundary conditions here, we omit this latter constitutive postulate.§ We therefore postulate

$$A^{(\alpha)} = A^{(\alpha)}\left(\frac{\partial x_i^{(1)}}{\partial X_j^{(1)}}, \frac{\partial x_i^{(2)}}{\partial X_j^{(2)}}, T^{(1)}, T^{(2)}\right), \qquad (\alpha = 1,2), \tag{3.1}$$

† See [9].

‡ For the definition of the terms in this expression, see [9].

§ If relative motion of the constituents occurs, as we have allowed here, then the boundary of the material may not consist of a heterogeneous mixture; the formulation of the correct boundary conditions then presents some difficulty. However, in the static problems considered later we have alleviated this difficulty by suitably choosing our experiments.

$$S^{(\alpha)} = S^{(\alpha)}\left(\frac{\partial x_i^{(1)}}{\partial X_j^{(1)}}, \frac{\partial x_i^{(2)}}{\partial X_j^{(2)}}, T^{(1)}, T^{(2)}\right), \qquad (\alpha = 1, 2), \tag{3.2}$$

$$\sigma_{(ik)}^{(1)} = A_{ik} + A_{ikj}(v_j^{(1)} - v_j^{(2)}); \qquad \sigma_{(ik)}^{(2)} = B_{ik} + B_{ikj}(v_j^{(1)} - v_j^{(2)}), \tag{3.3}$$

$$\beta_k^{(1)} = -\beta_k^{(2)} = -b_k - \beta_{kj}(v_j^{(1)} - v_j^{(2)}); \qquad \gamma_k^{(2)} = -\gamma_k^{(1)} = -c_k - \gamma_{kj}(v_j^{(1)} - v_j^{(2)}), \tag{3.4}$$

$$\beta_{ki}^{(1)} = -\beta_{ki}^{(2)} = -\beta'_{ki} - \beta'_{kij}(v_j^{(1)} - v_j^{(2)}); \qquad \gamma_{ki}^{(2)} = -\gamma_{ki}^{(1)} = -\gamma'_{ki} - \gamma'_{kij}(v_j^{(1)} - v_j^{(2)}), \tag{3.5}$$

$$\begin{aligned}\Psi^{(1)} = {} & \theta + \alpha_i(v_i^{(1)} - v_i^{(2)}) + \alpha_{ij}(v_i^{(1)} - v_i^{(2)})(v_j^{(1)} - v_j^{(2)}) + D_1 \frac{\mathrm{D}T^{(1)}}{\mathrm{D}t} + D_2 \frac{\mathrm{D}T^{(2)}}{\mathrm{D}t} \\ & + E_{ik}d_{ik}^{(1)} + E'_{ik}d_{ik}^{(2)} + E''_{ki}(\omega_{ik}^{(1)} - \omega_{ik}^{(2)}),\end{aligned} \tag{3.6}$$

$$\begin{aligned}\Psi^{(2)} = {} & -\theta - \alpha_i(v_i^{(1)} - v_i^{(2)}) - \alpha_{ij}(v_i^{(1)} - v_i^{(2)})(v_j^{(1)} - v_j^{(2)}) - D_1 \frac{\mathrm{D}T^{(1)}}{\mathrm{D}t} \\ & - D_2 \frac{\mathrm{D}T^{(2)}}{\mathrm{D}t} + G_{ik}d_{ik}^{(1)} + G'_{ik}d_{ik}^{(2)} + G''_{ki}(\omega_{ik}^{(1)} - \omega_{ik}^{(2)}),\end{aligned} \tag{3.7}$$

$$\Phi^{(1)} = \theta, \qquad \Phi^{(2)} = -\theta, \tag{3.8}$$

$$\lambda_{ki}^{(1)} = -\lambda_{ki}^{(2)} = \lambda_{ki} + \lambda_{kij}(v_j^{(1)} - v_j^{(2)}), \tag{3.9}$$

where the coefficients depend on the deformation gradients of each solid, the temperature of each and possibly upon the temperature of the mixture, and also obey suitable symmetry conditions,† e.g.

$$\beta'_{kij} = -\beta'_{ikj}.$$

In addition, $\beta_k^{(1)}$ and $\gamma_k^{(2)}$ may depend upon second deformation gradients of each solid. The temperature T of the mixture and the constitutive equations listed above will in general (for more general mixtures) depend upon the history of the individual temperatures and, perhaps, of some kinematical variables. Here, however, we restrict ourselves to consider constitutive equations which depend only on values of the temperatures $T^{(1)}$, $T^{(2)}$ and of the deformation gradients of the solids at time t. Some dependence upon the history is included by allowing the coefficients to depend upon T. We restrict ourselves to linear dependence upon $v_i^{(1)} - v_i^{(2)}$, except for the functions $\Psi^{(1)}$, $\Psi^{(2)}$.

Substituting from (3.1)–(3.6) and (3.8) into (2.11), and using (2.9) and the expressions for $\mathrm{D}F_{ij}^{(\alpha)}/\mathrm{D}t$, $(\alpha = 1, 2)$,‡ we obtain an inequality involving the variables $\mathrm{D}T^{(1)}/\mathrm{D}t$, $\mathrm{D}T^{(2)}/\mathrm{D}t$, $d_{ik}^{(1)}$, $d_{ik}^{(2)}$, $\omega_{ik}^{(1)} - \omega_{ik}^{(2)}$, $v_i^{(1)} - v_i^{(2)}$, $\omega_{ik}^{(1)} + \omega_{ik}^{(2)}$ and $T_{,k}$, the coefficients of which are independent of these quantities. By first choosing a temperature distribution for the mixture which is homogeneous, and then choosing arbitrary values of the remaining variables, we

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† See [9].

‡ See Crochet and Naghdi [12], equations (3.6).

can deduce that

$$\left.\begin{aligned}
S^{(1)} &= -\frac{\partial A^{(1)}}{\partial T^{(1)}}+\frac{D_1}{\rho^{(1)}}, \qquad D_2 = \rho^{(1)}\frac{\partial A^{(1)}}{\partial T^{(2)}},\\
A_{ik} &= \tfrac{1}{2}\rho^{(1)}\left(F^{(1)}_{kj}\frac{\partial A^{(1)}}{\partial F^{(1)}_{ij}}+F^{(1)}_{ij}\frac{\partial A^{(1)}}{\partial F^{(1)}_{kj}}\right)-E_{ik},\\
E'_{ik} &= \frac{1}{2}\rho^{(1)}\left(F^{(2)}_{kj}\frac{\partial A^{(1)}}{\partial F^{(2)}_{ij}}+F^{(2)}_{ij}\frac{\partial A^{(1)}}{\partial F^{(2)}_{kj}}\right),\\
\alpha_k+b_k &= \rho^{(1)}\left[\frac{\rho^{(1)}}{\rho}\frac{\partial A^{(1)}}{\partial F^{(2)}_{ij}}\{\operatorname{grad} F^{(2)}_{ij}\}_k-\frac{\rho^{(2)}}{\rho}\frac{\partial A^{(1)}}{\partial F^{(1)}_{ij}}\{\operatorname{grad} F^{(1)}_{ij}\}_k\right],\\
\beta'_{ki}+E''_{ki} &= \frac{1}{4}\rho^{(1)}\left\{\left(F^{(1)}_{kj}\frac{\partial A^{(1)}}{\partial F^{(1)}_{ij}}-F^{(1)}_{ij}\frac{\partial A^{(1)}}{\partial F^{(1)}_{kj}}\right)-\left(F^{(2)}_{kj}\frac{\partial A^{(1)}}{\partial F^{(2)}_{ij}}-F^{(2)}_{ij}\frac{\partial A^{(1)}}{\partial F^{(2)}_{kj}}\right)\right\},
\end{aligned}\right\} \tag{3.10}$$

$$\left(F^{(1)}_{kj}\frac{\partial A^{(1)}}{\partial F^{(1)}_{ij}}-F^{(1)}_{ij}\frac{\partial A^{(1)}}{\partial F^{(1)}_{kj}}\right)+\left(F^{(2)}_{kj}\frac{\partial A^{(1)}}{\partial F^{(2)}_{ij}}-F^{(2)}_{ij}\frac{\partial A^{(1)}}{\partial F^{(2)}_{kj}}\right) = 0, \tag{3.11}$$

$$A_{ikj} = 0, \beta'_{kij} = 0; \qquad (\beta_{ij}+\alpha_{ij})(v^{(1)}_i-v^{(2)}_i)(v^{(1)}_j-v^{(2)}_j)-\frac{\rho^{(1)}q_kT_{,k}}{\rho T} \geq 0. \tag{3.12}$$

Similarly, substituting (3.1)–(3.5), (3.7) and (3.8) into (2.12), we can deduce that

$$\left.\begin{aligned}
S^{(2)} &= -\frac{\partial A^{(2)}}{\partial T^{(2)}}-\frac{D_2}{\rho^{(2)}}, \qquad D_1 = -\rho^{(2)}\frac{\partial A^{(2)}}{\partial T^{(1)}},\\
B_{ik} &= \frac{1}{2}\rho^{(2)}\left(F^{(2)}_{kj}\frac{\partial A^{(2)}}{\partial F^{(2)}_{ij}}+F^{(2)}_{ij}\frac{\partial A^{(2)}}{\partial F^{(2)}_{kj}}\right)-G'_{ik},\\
G_{ik} &= \frac{1}{2}\rho^{(2)}\left(F^{(1)}_{kj}\frac{\partial A^{(2)}}{\partial F^{(1)}_{ij}}+F^{(1)}_{ij}\frac{\partial A^{(2)}}{\partial F^{(1)}_{kj}}\right),\\
-\alpha_k-c_k &= \rho^{(2)}\left[\frac{\rho^{(1)}}{\rho}\frac{\partial A^{(2)}}{\partial F^{(2)}_{ij}}\{\operatorname{grad} F^{(2)}_{ij}\}_k-\frac{\rho^{(2)}}{\rho}\frac{\partial A^{(2)}}{\partial F^{(1)}_{ij}}\{\operatorname{grad} F^{(1)}_{ij}\}_k\right],\\
-\gamma'_{ki}+G''_{ki} &= \frac{1}{4}\rho^{(2)}\left\{\left(F^{(1)}_{kj}\frac{\partial A^{(2)}}{\partial F^{(1)}_{ij}}-F^{(1)}_{ij}\frac{\partial A^{(2)}}{\partial F^{(1)}_{kj}}\right)-\left(F^{(2)}_{kj}\frac{\partial A^{(2)}}{\partial F^{(2)}_{ij}}-F^{(2)}_{ij}\frac{\partial A^{(2)}}{\partial F^{(2)}_{kj}}\right)\right\},
\end{aligned}\right\} \tag{3.13}$$

$$\left(F^{(1)}_{kj}\frac{\partial A^{(2)}}{\partial F^{(1)}_{ij}}-F^{(1)}_{ij}\frac{\partial A^{(2)}}{\partial F^{(1)}_{kj}}\right)+\left(F^{(2)}_{kj}\frac{\partial A^{(2)}}{\partial F^{(2)}_{ij}}-F^{(2)}_{ij}\frac{\partial A^{(2)}}{\partial F^{(2)}_{kj}}\right) = 0, \tag{3.14}$$

$$\beta_{ikj} = 0, \gamma'_{kij} = 0; \qquad (\gamma_{ij}+\alpha_{ij})(v^{(1)}_i-v^{(2)}_i)(v^{(1)}_j-v^{(2)}_j)+\frac{\rho^{(2)}q_kT_{,k}}{\rho T} \leq 0. \tag{3.15}$$

It can be shown, by considering invariance requirements under superposed rigid body motions, that (3.11) and (3.14) are identically satisfied, and using these equations we can rewrite $(3.10)_6$ and $(3.13)_6$ as

$$\left.\begin{aligned}
\beta'_{ki}+E''_{ki} &= \frac{1}{2}\rho^{(1)}\left(F^{(1)}_{kj}\frac{\partial A^{(1)}}{\partial F^{(1)}_{ij}}-F^{(1)}_{ij}\frac{\partial A^{(1)}}{\partial F^{(1)}_{kj}}\right),\\
-\gamma'_{ki}+G''_{ki} &= \frac{1}{2}\rho^{(2)}\left(F^{(1)}_{kj}\frac{\partial A^{(2)}}{\partial F^{(1)}_{ij}}-F^{(1)}_{ij}\frac{\partial A^{(2)}}{\partial F^{(1)}_{kj}}\right).
\end{aligned}\right. \tag{3.16}$$

Using (3.10), $(3.12)_{1,2}$, (3.13), $(3.15)_{1,2}$, and (3.16) in (2.15), and also using equation (2.13), we get

$$\rho\left(\frac{\mathrm{D}A}{\mathrm{D}t}+S\frac{\mathrm{D}T}{\mathrm{D}t}\right) = A_{ik}d_{ik}^{(1)}+B_{ik}d_{ik}^{(2)}-(\lambda_{ki}^{(1)}-\beta'_{ki}+\gamma'_{ki})(\omega_{ik}^{(1)}-\omega_{ik}^{(2)})$$

$$+\bar{\pi}_k^{(1)}(v_k^{(1)}-v_k^{(2)})-(\beta_{ki}-\gamma_{ki})(v_i^{(1)}-v_i^{(2)})(v_k^{(1)}-v_k^{(2)}). \qquad (3.17)$$

We now use an appropriate constitutive equation for $\bar{\pi}_k^{(1)}$, and equations (2.14) and (3.9) to recover the results (2.4)–(2.8) together with

$$D_{ki} = \beta'_{ki}-\gamma'_{ki}-\lambda_{ki}, \qquad \lambda_{kij} = 0, \qquad a_{kj} = \beta_{kj}-\gamma_{kj}. \qquad (3.18)$$

Also, using (3.8) in (2.16), we see that

$$\rho T\frac{\mathrm{D}S}{\mathrm{D}t} = \rho^{(1)}T^{(1)}\frac{\mathrm{D}S^{(1)}}{\mathrm{D}t}+\rho^{(2)}T^{(2)}\frac{\mathrm{D}S^{(2)}}{\mathrm{D}t}. \qquad (3.19)$$

The relation between the internal energy of the mixture and the internal energy of each constituent (Green and Naghdi [9], equation (4.40)) is satisfied when (3.17)–(3.19) are satisfied.

When the temperature distribution of each solid is the same, (3.19) reduces to

$$\rho\frac{\mathrm{D}S}{\mathrm{D}t} = \rho^{(1)}\frac{\mathrm{D}S^{(1)}}{\mathrm{D}t}+\rho^{(2)}\frac{\mathrm{D}S^{(2)}}{\mathrm{D}t}, \qquad (3.20)$$

and we can relate the entropy of the mixture to the entropy of each solid when in the mixture. Alternatively, if a relation between the entropy functions of each solid and that of the mixture is assumed, then (3.19) gives a relation between the temperature of the mixture and the temperature of each solid.

4. LINEARIZED THEORY OF TWO ISOTROPIC ELASTIC SOLIDS

We now consider a mixture of two isotropic elastic solids which are subject to small deformations and temperature changes. We shall suppose in the following that instead of occupying the same final position, the particles of each solid under consideration occupy the same position initially; however, since the displacements of each solid are assumed small, the theories of Green and Naghdi ([8] and [9]) will still hold. The displacements of each solid, the temperature changes of each and their space and time derivatives are assumed small and squares of these quantities are neglected in the expressions for the partial stresses, diffusive force, etc. A comma now denotes spatial differentiation with respect to the initial position of each solid; to our order of approximation this can usually equally well be considered as spatial differentiation with respect to the final position of either solid.

Because of invariance requirements under superposed rigid body motions and because of the isotropy of each solid, we can write the free energy functions in the form†

$$A = A(F_{im}^{(1)}F_{jm}^{(1)}, F_{rn}^{(2)}F_{sn}^{(2)}, T), \qquad A_\alpha = A_\alpha(F_{im}^{(1)}F_{jm}^{(1)}, F_{rn}^{(2)}F_{sn}^{(2)}, T_1, T_2), \qquad (\alpha = 1, 2),$$

† This follows in a similar manner to that done for a single isotropic elastic solid, see [10] for details; in this and subsequent sections we write $A_1, A_2, T_1, T_2, \rho_1, \rho_2 \ldots$ in place of $A^{(1)}, A^{(2)}, T^{(1)}, T^{(2)}, \rho^{(1)}, \rho^{(2)} \ldots$ and we also try to be consistent with the notation used previously (see, e.g. [13]).

where A, A_1, A_2 are isotropic functions of their arguments. Since the displacements and temperature changes are small, we can therefore write

$$\left.\begin{aligned}
\bar{\rho}A &= \bar{\rho}\bar{A}+\alpha_1 e_{mm}+\alpha_2 g_{mm}+\alpha_3\theta+\alpha_5 e_{rs}e_{rs}+\alpha_6 g_{rs}g_{rs}\\
&\quad+\alpha_7 e_{rs}g_{rs}+\tfrac{1}{2}\alpha_8 e_{rr}e_{mm}+\tfrac{1}{2}\alpha_9 g_{rr}g_{mm}+\tfrac{1}{2}\alpha_{10}\theta^2\\
&\quad+\alpha_{12}e_{rr}g_{mm}+\alpha_{13}e_{mm}\theta+\alpha_{15}g_{mm}\theta,\\
\bar{\rho}_1 A_1 &= \bar{\rho}_1\bar{A}_1+\beta_1 e_{mm}+\beta_2 g_{mm}+\beta_3\theta_1+\beta_4\theta_2+\beta_5 e_{rs}e_{rs}\\
&\quad+\beta_6 g_{rs}g_{rs}+\beta_7 e_{rs}g_{rs}+\tfrac{1}{2}\beta_8 e_{rr}e_{mm}+\tfrac{1}{2}\beta_9 g_{rr}g_{mm}\\
&\quad+\tfrac{1}{2}\beta_{10}\theta_1^2+\tfrac{1}{2}\beta_{11}\theta_2^2+\beta_{12}e_{rr}g_{mm}+\beta_{13}e_{mm}\theta_1+\beta_{14}e_{mm}\theta_2\\
&\quad+\beta_{15}g_{mm}\theta_1+\beta_{16}g_{mm}\theta_2+\beta_{17}\theta_1\theta_2,\\
\bar{\rho}_2 A_2 &= \bar{\rho}_2\bar{A}_2+\gamma_1 e_{mm}+\gamma_2 g_{mm}+\gamma_3\theta_1+\gamma_4\theta_2+\gamma_5 e_{rs}e_{rs}\\
&\quad+\gamma_6 g_{rs}g_{rs}+\gamma_7 e_{rs}g_{rs}+\tfrac{1}{2}\gamma_8 e_{rr}e_{mm}+\tfrac{1}{2}\gamma_9 g_{rr}g_{mm}+\tfrac{1}{2}\gamma_{10}\theta_1^2\\
&\quad+\tfrac{1}{2}\gamma_{11}\theta_2^2+\gamma_{12}e_{rr}g_{mm}+\gamma_{13}e_{mm}\theta_1+\gamma_{14}e_{mm}\theta_2+\gamma_{15}g_{mm}\theta_1\\
&\quad+\gamma_{16}g_{mm}\theta_2+\gamma_{17}\theta_1\theta_2,
\end{aligned}\right\} \tag{4.1}$$

where e_{rs}, g_{rs} are the usual linearized strain tensors for solid 1 and solid 2 respectively,† θ_1 and θ_2 are the temperature changes of the two particles under consideration and a bar denotes the value of the variable in the initial position. The coefficients are material constants (which depend on the initial values of the densities and temperatures, assumed homogeneous).

From (2.5)–(2.8), using (4.1), we have

$$\left.\begin{aligned}
\sigma^{(1)}_{(ik)} &= \alpha_1\delta_{ik}+2(\alpha_1+\alpha_5)e_{ik}+\alpha_7 g_{ik}+\left(\alpha_8-\frac{\bar{\rho}_1}{\bar{\rho}}\alpha_1\right)e_{mm}\delta_{ik}\\
&\quad+\left(\alpha_{12}-\frac{\bar{\rho}_2}{\bar{\rho}}\alpha_1\right)g_{mm}\delta_{ik}+\alpha_{13}\theta\delta_{ik},\\
\sigma^{(2)}_{(ik)} &= \alpha_2\delta_{ik}+2(\alpha_2+\alpha_6)g_{ik}+\alpha_7 e_{ik}+\left(\alpha_9-\frac{\bar{\rho}_2}{\bar{\rho}}\alpha_2\right)g_{mm}\delta_{ik}\\
&\quad+\left(\alpha_{12}-\frac{\bar{\rho}_1}{\bar{\rho}}\alpha_2\right)e_{mm}\delta_{ik}+\alpha_{15}\theta\delta_{ik},\\
\sigma^{(1)}_{[ki]} &= -\sigma^{(2)}_{[ki]} = 0,\\
\bar{\pi}^{(1)}_k &= \frac{\bar{\rho}_1}{\bar{\rho}}\alpha_2 g_{mm,k}-\frac{\bar{\rho}_2}{\bar{\rho}}\alpha_1 e_{mm,k}+\alpha(v^{(1)}_k-v^{(2)}_k),
\end{aligned}\right\} \tag{4.2}$$

where α is another material constant.

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† For the definition of these, see [13], equations (4.4) and (8.3).

Also, if we now suppose that each solid has the same temperature distribution, then from (3.10), (3.13) and (3.20), using (4.1), we find that

$$\alpha_{10}\frac{\partial\theta}{\partial t}+\alpha_{13}\frac{\partial e_{mm}}{\partial t}+\alpha_{15}\frac{\partial g_{mm}}{\partial t} = \{\beta_{10}+\gamma_{10}+\beta_{11}+\gamma_{11}+2(\beta_{17}+\gamma_{17})\}\frac{\partial\theta}{\partial t}$$
$$+(\beta_{13}+\gamma_{13}+\beta_{14}+\gamma_{14}+\gamma_3-\beta_4)\frac{\partial e_{mm}}{\partial t}+(\beta_{15}+\gamma_{15}+\beta_{16}+\gamma_{16}-\gamma_3+\beta_4)\frac{\partial g_{mm}}{\partial t}.$$

We can therefore deduce that

$$\alpha_{10} = \beta_{10}+\gamma_{10}+\beta_{11}+\gamma_{11}+2(\beta_{17}+\gamma_{17}),\qquad \alpha_{13} = \beta_{13}+\gamma_{13}+\beta_{14}+\gamma_{14}+\gamma_3-\beta_4,$$
$$\alpha_{15} = \beta_{15}+\gamma_{15}+\beta_{16}+\gamma_{16}-\gamma_3+\beta_4. \tag{4.3}$$

5. DETERMINATION OF THE STRESS–STRAIN COEFFICIENTS

We now assume that in the initial position there are no partial stresses, so that (4.2) yields[†]

$$\begin{aligned}\sigma_{ik}^{(1)} = \sigma_{ki}^{(1)} &= \lambda_1 e_{mm}\delta_{ik}+2\mu_1 e_{ik}+\lambda_3 g_{mm}\delta_{ik}+2\mu_3 g_{ik}+\alpha_{13}\theta\delta_{ik},\\ \sigma_{ik}^{(2)} = \sigma_{ki}^{(2)} &= \lambda_2 g_{mm}\delta_{ik}+2\mu_2 g_{ik}+\lambda_3 e_{mm}\delta_{ik}+2\mu_3 e_{ik}+\alpha_{15}\theta\delta_{ik},\end{aligned} \tag{5.1}$$

where λ_1, μ_1, etc. are suitably defined in terms of α_5, α_6, etc.

We further assume that the only interaction terms in these relations are those involving λ_3 and μ_3; thus λ_1, μ_1 and λ_2, μ_2 are the Lamé elastic constants for solids 1 and 2 respectively when separated, and if $\varkappa_1 = \lambda_1+\frac{2}{3}\mu_1$, $\varkappa_2 = \lambda_2+\frac{2}{3}\mu_2$ are the bulk moduli of each solid, then

$$\alpha_{13} = -\sigma_1\varkappa_1,\qquad \alpha_{15} = -\sigma_2\varkappa_2, \tag{5.2}$$

where σ_1 and σ_2 are the volume coefficients of thermal expansion for solids 1 and 2 respectively. It can be seen from (4.1) and (4.3) that these assumptions are consistent with, e.g., the assumption that the free energy of each solid is unchanged on mixing, but allow more generality.

Other authors have made similar assumptions, usually of a more restrictive nature. In the literature on composite materials it is customary to assume relations of the form (5.1) in the absence of thermal effects, exclude interaction so that $\lambda_3 = \mu_3 = 0$, consider a mean strain tensor which in the notation of this paper is defined through

$$\bar{\rho}\varepsilon_{ij} = \bar{\rho}_1 e_{ij}+\bar{\rho}_2 g_{ij},$$

and, in addition, make further assumptions regarding relationships between the "partial strains" and the mean strain ε_{ij}. Such additional assumptions, if acceptable or applicable in particular applications, will enable one to write an expression for the total stress of the form

$$t_{ij} = \lambda'\varepsilon_{kk}\delta_{ij}+2\mu'\varepsilon_{ij}, \tag{5.3}$$

[†] Notice that although the diffusive force $\bar{\pi}_k^{(1)}$ is now zero in static problems, there is still mechanical interaction between the solids due to the interaction terms in the partial stresses. Furthermore, the diffusive force will be non-zero in dynamical problems.

where λ', μ' are related to $\lambda_1, \lambda_2, \mu_1, \mu_2, \bar{\rho}_1, \bar{\rho}_2$ and coefficients which are brought in through the relationships between ε_{ij} and the "partial strains" mentioned above.

A number of authors, using results of this form for the isothermal case, have proposed bounds on the "overall" coefficients λ' and μ'. Here, it is not our purpose to follow the line of argument pursued previously in the literature; instead in the rest of this section we confine our attention toward a more direct evaluation of the coefficients which occur in (5.1).

Under the assumptions previous to (5.2) then, we examine the material coefficients appearing in (5.1).

In the experiments which follow there may be difficulties if the moduli of each solid were to change on mixing. However, we have explicitly assumed here that the elastic and thermal moduli of the solids when mixed are the same as when separate, the only difference when mixed being the addition of interaction terms. The experiment involving thermal expansion requires the measurement of the change in volume of the mixture, which may not be possible; however, the mean displacement of the mixture as used in this paper is exactly that which is assumed in the theory of composite materials to be the actual displacement observed when deforming an alloy (see, e.g. Hill [2]) hence the term "overall" moduli used in these theories.

If we raise the temperature of a unit of volume of the mixture by one degree while keeping it free of applied stresses, then by definition the volume change of the mixture (per unit volume) is equal to the volume coefficient of thermal expansion for the mixture. Since the total mass is conserved, the change in volume of the mixture per unit volume is equal and opposite to the change in density of the mixture per unit mass, to our order of approximation. Now

$$\begin{aligned} \rho &= \bar{\rho} - \bar{\rho}_1 e_{mm} - \bar{\rho}_2 g_{mm},\dagger \\ &= \bar{\rho}(1 - \varepsilon_{mm}), \end{aligned} \tag{5.4}$$

where ε_{ik} is the mean strain tensor.

From (5.1) when there are no applied stresses and the temperature is raised by one degree, we have‡

$$\begin{aligned} \kappa_1 e_{mm} + \kappa_3 g_{mm} + \alpha_{13} &= 0, \\ \kappa_2 g_{mm} + \kappa_3 e_{mm} + \alpha_{15} &= 0, \end{aligned} \tag{5.5}$$

where $\kappa_3 = \lambda_3 + \frac{2}{3}\mu_3$. Therefore, using (5.2)–(5.5), the volume coefficient of thermal expansion for the mixture, σ, is given by

$$\sigma = \frac{1}{\bar{\rho}(\kappa_1\kappa_2 - \kappa_3^2)}\{\sigma_1\kappa_1(\bar{\rho}_1\kappa_2 - \bar{\rho}_2\kappa_3) + \sigma_2\kappa_2(\bar{\rho}_2\kappa_1 - \bar{\rho}_1\kappa_3)\}. \tag{5.6}$$

If we can determine the volume coefficient of thermal expansion for the mixture by experiment, then knowing the properties of each solid we can calculate κ_3.

We now consider a further theoretical experiment, viz., the torsion of a cylinder consisting of the above mixture. We choose cylindrical polar co-ordinate axes in the initial body so that the z-axis lies along the axis of the cylinder, and so that the plane ends of the cylinder are at $z = \pm l$. The cylinder has a circular cross-section of radius a and we apply

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† See [13], equation (8.4).

‡ Note that similar equations (with different coefficients) still hold if the initial partial stresses are non-zero, since the applied stresses are those over and above the initial stresses.

equal and opposite couples at the plane ends by binding them to rigid plates and twisting. We hold the cylinder in equilibrium keeping the curved surfaces stress-free. The deformation is assumed to be pure torsion and the temperature remains constant throughout. Denoting the cylindrical polar co-ordinates by (r, ϕ, z), the initial and final co-ordinates of the particles under consideration are given by

$$\begin{aligned} &X_1^{(1)} = X_1^{(2)} = r\cos\phi, && X_2^{(1)} = X_2^{(2)} = r\sin\phi, && X_3^{(1)} = X_3^{(2)} = z, \\ &x_1^{(1)} = r\cos(\phi+\psi_1 z), && x_2^{(1)} = r\sin(\phi+\psi_1 z), && x_3^{(1)} = z, \\ &x_1^{(2)} = r\cos(\phi+\psi_2 z), && x_2^{(2)} = r\sin(\phi+\psi_2 z), && x_3^{(2)} = z. \end{aligned} \tag{5.7}$$

Examination of (4.2) and the equations of equilibrium with no body forces† (or the corresponding equations in curvilinear co-ordinates, see [10]), using (5.7), reveals that ψ_1 and ψ_2 must be constants. Further the only non-zero physical components of stress are

$$\begin{aligned} \sigma_{z\phi}^{(1)} &= (\mu_1\psi_1+\mu_3\psi_2)r, & \sigma_{z\phi}^{(2)} &= (\mu_2\psi_2+\mu_3\psi_1)r, \\ &= \sigma_{\phi z}^{(1)}, & &= \sigma_{\phi z}^{(2)}. \end{aligned} \tag{5.8}$$

We assume that since the ends $z = \pm l$ are fixed to rigid plates, the displacement of each solid will be the same at these ends. From (5.7) we therefore see that $\psi_1 = \psi_2 = \psi$ say.

The total‡ couple at each end of the cylinder is therefore given by

$$\begin{aligned} M &= 2\pi\int_0^a (\sigma_{z\phi}^{(1)}+\sigma_{z\phi}^{(2)})r^2\,\mathrm{d}r, \\ &= \frac{\pi a^4(\mu_1+\mu_2+2\mu_3)\psi}{2}, \end{aligned} \tag{5.9}$$

using (5.8).

If we measure the angle of twist at the ends $z = \pm l$ of the cylinder in radians ($= \psi l$), and also measure the total couple at these ends needed to maintain this angle, then we can calculate the constant μ_3.

By means of two theoretically simple experiments, therefore, we can determine all three required material constants for the mixture, viz., the volume coefficient of thermal expansion and the constants λ_3, μ_3.

Because of the nature of our assumptions at the start of this section, we see from (4.2) that the diffusive force will be zero in equilibrium for the type of mixture we are considering. If this state of affairs is not desirable, we can return to equations (4.2) instead of using (5.1) and redefine constants $\lambda_1, \mu_1, \lambda_2, \mu_2, \lambda_3, \lambda_4, \mu_3$. We can then use the fact that the total stress is zero initially to deduce that $\alpha_2 = -\alpha_1$§, and therefore that $\lambda_4-\lambda_3 = \alpha_1$. We can therefore use the above two experiments in conjunction with one other experiment (for example, compression at constant temperature with lateral expansion prevented) to determine the initial partial stress, the volume coefficient of thermal expansion for the mixture and the constants λ_3, λ_4 and μ_3. In this case we assume that α_5, α_6, α_8 and α_9 in (4.2) are the original Lamé constants for each solid when separate, and the constants $\lambda_1, \lambda_2, \mu_1$ and μ_2 are suitably defined.

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† See [7], equation (6.2).

‡ The definition of total stress used here is that given by Green and Naghdi ([8] and [9]). Hill [2], who considers a mixture of two solids from a slightly different viewpoint, has a different definition of total stress.

§ Cf. [13], equation (8.12).

In conclusion we remark briefly on the concept of effective moduli for the mixture. These are the moduli appearing in the relation between total stress and some kind of total strain, and previous authors have attempted to evaluate them or have evaluated bounds for them (see, e.g. [1]–[4]). The difficulty is in obtaining a linear relation between total stress and total strain. This can be done by either taking a "representative volume" (see Hill [2]) and assuming that the actual strains of each solid are uniquely related to the total strain, or by assuming, for example, that the total surface force in equilibrium is uniquely divided between each solid. Both approaches are essentially equivalent and enable one to obtain the desired linear relationship, which will include the coefficients describing the division of total strain or total surface force.

The concept of the total surface force being divided uniquely between the two solids because of the nature of the surface of the mixture (bearing in mind that the mixture is homogeneous initially) is an attractive one and apparently physically reasonable. However, if we assume that in any mixture we can restrict the displacements of each solid to be equal at the boundary by suitably applying boundary conditions (in other words construct a particular displacement-boundary-value problem, also apparently reasonable), as has been assumed already in this paper, then we are led to the conclusion that the coefficients describing the division of total surface force (or total strain) have certain values independent of the nature of the mixture.

To resolve this apparent paradox we suggest that the division of total surface force may depend upon the nature of the boundary conditions.

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REFERENCES

[1] Z. HASHIN and S. SHTRIKMAN, *J. Mech. Phys. Solids* **11**, 127 (1963).
[2] R. HILL, *J. Mech. Phys. Solids* **11**, 357 (1963).
[3] L. J. WALPOLE, *J. Mech. Phys. Solids* **14**, 151 (1966).
[4] L. J. WALPOLE, *J. Mech. Phys. Solids* **14**, 289 (1966).
[5] P. G. SHEWMON, *Diffusion in Solids*. McGraw-Hill (1963).
[6] W. JOST, *Diffusion*, Chapter 8. Academic Press (1952).
[7] A. E. GREEN and T. R. STEEL, *Int. J. engng Sci.* **4**, 483 (1966).
[8] A. E. GREEN and P. M. NAGHDI, *Int. J. engng Sci.* **3**, 231 (1965).
[9] A. E. GREEN and P. M. NAGHDI, *Archs ration. Mech. Analysis* **24**, 243 (1967).
[10] N. MILLS and T. R. STEEL, to be published.
[11] A. E. GREEN and P. M. NAGHDI, *Int. J. engng Sci.* To be published.
[12] M. J. CROCHET and P. M. NAGHDI, *Int. J. engng Sci.* **4**, 383 (1966).
[13] T. R. STEEL, *Q. Jl Mech. appl. Math.* **20**, 57 (1967).

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Абстракт—Исполвзуя теорию смесей, допускающих разные темиературы для каждого составного злемента, обсуждаются сперва нелинейные определяющие уравнения для смесы, состоящей из двух упругих твердых тел. Затем, в качестве специального случая, определяютсня завимотси между некоторыми постоянными материала, отнесенными к каждому составному злементу и соответствующими постоянными, отнесеными к смеси. Линиаризуются определяющие уравнения, принмая во внимание, что все перемещения недеформированного состояния и все последующие изменения температуры—малы. Далее показано как оценить постоянные, появляющиеся в линиаризованных определяющих уравнениях, при некоторых предположениях.